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# EVOLUTION OF AXISYMMETRIC VORTICITY DISTRIBUTIONS IN AN IDEAL INCOMPRESSIBLE STRATIFIED LIQUID*

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The Cauchy problem is solved for axisymmetric vortex perturbations of an exponentially stratified incompressible ideal liquid. The behaviour of vorticity inside the region of its initial location, near the boundary of that region, and away from it in the "wave" zone is studied. A number of examples and analyzed with a specific initial distribution of vorticity, among which are examples of anomalous solution behaviour. It is shown that the initial jump of vorticity in a stratified liquid does not vanish, but oscillates at a frequency which depends on the direction. When passing to the limit of strongly singular initial distributions of the vortex filament and cylindrical vortex layer types, the solution increases with time.

In liquids of inhomogeneous density (stratified liquids) in a gravity field the Archimedes forces in the absence of a free surface sustain the propagation of interval waves. Such waves have a vortex character and are capable of "carrying away" the vorticity from its original location region. Because of this, the vorticity distributions which were stationary in a homogeneous liquid may become unsteady in a stratified liquid, even when viscosity is neglected.

Below, we consider, in the linear approximation of an ideal incompressible inhomogeneous liquid, the evolution of initially axisymmetric vorticity distributions (with a horizontal axis of symmetry). The exact solutions obtained are in many respects close to the solution of the problem of the "collapse" of a cylindrical region with initial density perturbations, obtained earlier /1/. This similarity is associated with the fact that the vortex motion results in violation of the initial equilibrium density distribution in some region which subsequently leads to its "collapse". The solution of the general problem of the collapse of a mixed liquid, allowing for the initial density perturbations and vorticity in the linear description can be represented by the sum of solutions of problems with initial perturbations of only one of these characteristics, i.e. of the solutions considered below and solutions similar to that obtained in /1/.

1. As the basic perturbed state let us consider a stationary liquid with an exponential density distribution along the vertical $\exp(N^2z/g)$. The equations of small two-dimensional perturbations (we restrict the consideration to two-dimensional motions in the vertical plane) in the linear approximation of an ideal incompressible liquid can be then written in the form (e.g., /2/)

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}\quad \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \rho g,\quad \frac{\partial \rho}{\partial t} + \frac{N^2}{g} w = 0,$$
$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

where $p, \rho, u, w$ are perturbations of the pressure, density and velocity components, $x, z, t$ are space coordinates ($z > 0$ downwards) and time, and $g$ and $N$ are the free-fall acceleration and the buoyancy frequency. The Boussinesq approximation is used here according to which the variation of the density in inertial terms is not taken into account.

The evolution of the initial verticity distribution $\omega = \partial u/\partial z - \partial w/\partial x$ is defined by the equation

$$L\omega(\mathbf{r}, t) = 0,\quad L \equiv \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) + N^2\frac{\partial^2}{\partial x^2} \tag{1.1}$$

which follows from the above system, and by the initial conditions

$$\omega|_{t=0} = \omega(\mathbf{r}, 0),\quad \left.\frac{\partial \omega}{\partial t}\right|_{t=0} = 0 \tag{1.2}$$

The second of these conditions indicates the absence of initial density perturbations (the original system of equations implies that $\partial\omega/\partial t = -g\,\partial\rho/\partial x$).

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Using a Fourier transformation, the solutions of such an initial value problem can be represented in the form

$$\omega(\mathbf{r}, t) = \frac{1}{(2\pi)^3} \int d^3k \omega(\mathbf{k}, 0) \cos\left(Nt \frac{k_x}{k}\right) \exp(i\mathbf{kr}) \tag{1.3}$$

where, as is clear from the special case $t = 0$, $\omega(\mathbf{k}, 0)$ is the Fourier transform of the initial vorticity distribution $\omega(\mathbf{r}, 0)$.

Restricting the analysis of axisymmetric initial vorticity distributions it is possible to reduce the number of integrations in (1.3), since $\omega(\mathbf{k}, 0)$ then actually depends only on the magnitude of the wave vector $k = |\mathbf{k}|$. After integration with respect to the directions of the vector $k$, we obtain the following integral representation of the solution in terms of the Bessel function /3/:

$$\omega(\mathbf{r}, t) = \frac{1}{4\pi} \int_0^\infty dkk\omega(k, 0) \sum_\pm J_0(R_\pm) \tag{1.4}$$

$$R_\pm^2 = (kr \pm Nt |\cos\varphi|)^2 + N^2t^2 \sin^2\varphi, \quad x = r\cos\varphi, \quad z = r\sin\varphi$$

(certain singularities of similar representations were considered in /4/).

By expanding this expression in a Fourier series in the angular variable $\varphi$, which can be done using the theorem of the summation of cylindrical functions /3/, then in view of the angular symmetry $(\varphi \to \pi - \varphi, \ \varphi \to -\varphi)$, only harmonies of even number and angle are present in the expansion

$$\omega(\mathbf{r}, t) = \sum_{n=0}^\infty \varepsilon_n a_{2n}(r) J_{2n}(Nt) \cos 2n\varphi \tag{1.5}$$

$$a_{2n}(r) = \frac{1}{2\pi} \int_0^\infty dkk\omega(k, 0) J_{2n}(kr)$$

Such a Fourier series is at the same time a Neumann expansion in cylindrical functions.

When $r = 0$ or when averaging over directions, the whole series reduces to the first term

$$\omega(\mathbf{r}, t)|_{r=0} = \omega(0, 0) J_0(Nt), \quad \langle \omega(\mathbf{r}, t) \rangle = \omega(r, 0) J_0(Nt) \tag{1.6}$$

so that for the vorticity averaged over the vorticity angles $\langle \omega(\mathbf{r}, t) \rangle$ and for the vorticity on the axis $\omega(0, t)$, a characteristic feature is oscillating damping (if of course, $\omega(0, 0) \neq 0$, cf. Sect.2, example 2). For long times $(Nt \gg 1)$ in conformity with the asymptotic behaviour of the Bessel function, the oscillations of the buoyancy fequency $N$ are damped as $\sim (Nt)^{-1/2}$.

A simple, but unexpected result follows for the total magnitude of the vorticity over the radial half-line (assuming it to be finite)

$$\int_0^\infty dr\omega(\mathbf{r}, t) = \cos(Nt \sin\varphi) \int_0^\infty dr\omega(r, 0)$$

(when integrating (1.5) it is necessary to use the well-known formulas /3/).

Thus in the linear approximation of an ideal liquid the total vorticity on the radial half-line does not decrease with time but oscillates at a frequency dependent on the direction. In the horizontal direction it generally does not vary.

The vorticity, which initially was concentrated in a region with characteristic scale $r_0$, is carried away by internal waves for a considerable time far beyond the boundaries of the region. The integral (1.4) then yields a simple asymptotic estimate. If $Nt|\sin\varphi| \gg 1$, the Bessel function under the integral can be replaced by a trigonometric asymptotic form. On the assumption that $r \gg r_0$ the function $\omega(k, 0)$ is found to vary much more slowly than the trigonometric one, and the stationary points (using the method of stationary phase, or that of steepest descent) can be determined using the argument of the latter. When

$$\varphi \neq \pm \frac{\pi}{2}$$

from two such stationary points only the point $k_c = Nt|\cos\varphi|/r$ falls in the integration interval, and the estimate of its contribution in its nearest neighbourhood yields

$$\omega(\mathbf{r}, t) \approx \cos(Nt \sin\varphi) \frac{k_c \omega(k_c, 0)}{2\pi r}, \quad k_c = \frac{Nt}{r}|\cos\varphi| \tag{1.7}$$

The general solution analyzed above can also be expressed in terms of the integral of the initial three-dimensional vorticity distribution, using the inverse Fourier transform from $\omega(\mathbf{k}, 0)$ to $\omega(\mathbf{r}, 0)$. In the case of an axisymmetric distribution, the two-dimensional Fourier transform reduces to the Hankel transform, using which, and also the theorem of the summation of cylindrical functions /3/ and formulas for the integrals of products of Bessel functions /3/, it is possible to derive from (1.4)

$$\omega(\mathbf{r},t)=\omega(0,0)J_0(Nt)+\int_0^r dr'\,\frac{\partial\omega(r',0)}{\partial r'}\,K\left(Nt,\frac{r'}{r},\varphi\right) \tag{1.8}$$

$$K\left(Nt,\frac{r'}{r},\varphi\right)=J_0(Nt)-\frac{r'}{r}\int_0^\infty dkJ_1(kr')\sum_{\pm}J_0(R_\pm)= \tag{1.9}$$

$$J_0(Nt)+\sum_{n=1}^\infty\{P_n(\xi)-P_{n-1}(\xi)\}J_{2n}(Nt)\cos 2n\varphi,\quad r'\leqslant r$$

where, since the function $K\left(Nt,\frac{r'}{r},\varphi\right)$ vanishes when $r'>r$, in the integral in (1.8) we insert a finite upper limit, and $P_n(\xi)$ is the Legendre polynomial in $\xi=1-2\frac{r'^2}{r^2}$ which can be replaced by the Jacobi polynomial $P_n^{(1,0)}(\xi)=(P_n(\xi)-P_{n+1}(\xi))/(1-\xi)$.

Let us give one more integral representation of the function $K\left(Nt,\frac{r'}{r},\varphi\right)$, which can be obtained from (1.9), using the well-known formulas /3/ and a certain substitution of the angle variable of integration

$$K\left(Nt,\frac{r'}{r},\varphi\right)=\frac{2}{\pi}\int_0^{\pi/2}d\alpha\cos(a\sin\alpha)\cos\left(b\left(\cos^2\alpha+\frac{r'^2}{r^2}\sin^2\alpha\right)^{1/2}\right)$$

$$a=Nt\sin\varphi\left(1-\frac{r'^2}{r^2}\right)^{1/2},\quad b=Nt\cos\varphi,\quad r'<r$$

which for the vertical direction ($\varphi=\pi/2$) is in fact the integral representation of the Bessel function

$$K\left(Nt,\frac{r'}{r},\varphi\right)=J_0\left(Nt\left(1-\frac{r'^2}{r^2}\right)^{1/2}\right),\quad |z|>r' \tag{1.10}$$

Note that the possibility of such a simple result for the integral of the products of Bessel functions from (1.9), when $\varphi=\frac{\pi}{2}$ and $r>r'$ had previously given rise to doubts /5/, and is not given in /3/. Only in /6/ was a method of calculating a fairly large class of such integrals proposed.

For the horizontal direction ($\varphi=0$) the integral is reduced to the derivative of an integral proportional to the three-dimensional Green's function of the internal-waves operator $L$ whose behaviour is investigated in detail in /7,8/(*).

$$K\left(Nt,\frac{r'}{r},0\right)=\frac{2}{\pi}\frac{\partial}{\partial t}\int_{Nr'/r}^{N}d\omega\left[(N^2-\omega^2)\left(\omega^2-N^2\frac{r'^2}{r^2}\right)\right]^{1/2}\sin\omega t \tag{1.11}$$

2. Let us consider some examples, firstly, of the "vortex filament", when the vorticity initially uniformly fills a cylinder of radius $r_0$.

*Example 1.* For the distribution of the vortex filament type ($h(r_0-r)$ is the Heaviside unit function)

$$\omega(r,0)=\omega_0h(r_0-r),\quad \omega(k,0)=2\pi r_0\omega_0J_1(kr_0)/k$$

and from the general formulas (1.4) or (1.8), (1.9) the vorticity evolution is defined by the integral

$$\omega(\mathbf{r},t)=\frac{\omega_0r_0}{2}\int_0^\infty dkJ_1(kr_0)\sum_{\pm}J_0(R_\pm) \tag{2.1}$$

which inside the region initially filled by vortices reduces to the Bessel function

$$\omega(r,t)=\omega_0J_0(Nt),\quad r<r_0 \tag{2.2}$$

i.e. the vorticity there oscillates and is damped uniformly for all $r<r_0$ (campare with (1.6)). In the region outside the boundaries of the original filament $r\geqslant r_0$ the result may be represented in the form of a Neumann series (see (1.8) and (1.9))

$$\frac{\omega}{\omega_0}=\sum_{n=1}^\infty\left\{P_{n-1}\left(1-\frac{2r_0^2}{r^2}\right)-P_n\left(1-\frac{2r_0^2}{r^2}\right)\right\}J_{2n}(Nt)\cos 2n\varphi,\quad r\geqslant r_0 \tag{2.3}$$

*)See /7,8/ and Gorodtsov V.A. and Teodorovich E.B., Linear internal waves in an exponentially stratified ideal incompressible liquid. Pre-print No.114, Inst. Problems Mekhan., Acad. Nauk SSSR, 1978; and Gorodtsov V.A. and Teodorovich E.V., The Cherenkov radiation of internal waves by a uniformly moving source. Preprint No.183, Inst. Problem Mekhan., AN SSSR, 1981.

Summing this series for $r = r_0$ and using the relation $P_n(-1) = (-1)^n$ and the formulas for the expansion of a trigonometric function in Bessel functions /3/, for the jump at $r \equiv r_0$ we obtain

$$[\omega(\mathbf{r}, t)] = -\omega_0 \cos(Nt \cos\varphi), \quad r = r_0 \qquad (2.4)$$

that indicates that the original vorticity jump at the boundary of the vortex filament does not vanish in time but oscillates at frequency dependent on the direction.

Retaining in (2.3) terms linear with respect to the small ratio $(r - r_0)/r_0$, we obtain

$$\omega \approx \omega_0\left(1 - \frac{r - r_0}{2r_0}\frac{\partial^2}{\partial\varphi^2} + \ldots\right)\{J_0(Nt) - \cos(Nt\cos\varphi)\}, \quad r > r_0$$

which makes clear that the jump zone decreases rapidly. The smallness of the correction is here related to the smallness of the parameter $N^2t^2(r - r_0)/r_0$.

For the vertical direction, by virtue of (1.8) and (1.10), we have the formula

$$\omega\big|_{\varphi=\frac{\pi}{2}} = \omega_0 J_0(Nt) - \omega_0 J_0\left(Nt\sqrt{1 - r_0^2/z^2}\right) h(|z| - r_0) \qquad (2.5)$$

which confirms the previous results.

A fairly simple analysis is possible for the horizontal direction owing to the direct relation of the answer to Green's function (see (1.11)).

Finally, in the far "wave" zone at $Nt\,|\sin\varphi| \gg 1$, $r/r_0 \gg 1$ from (1.7) we have the asymptotic form

$$\omega(r, t) \approx \omega_0 \frac{r_0}{r} J_1\left(Nt\frac{r_0}{r}|\cos\varphi|\right)\cos(Nt\sin\varphi)$$

Note that a similar analysis using the theorem of summation and recurrent formulas for cylindrical functions, is possible for the more general integrals of $J_l(kr_0)k^m J_n(R_\pm)R_\pm^{-n}$. The case when $n = m = 1$, $l = 2$ was considered in /1/.

The singularity of the above solution lies in the fact that it conserves the vorticity shock amplitude (see also /1/). This is evidently related to the disregard of viscosity, and the non-linearity in the formulation of the input problem. Consideration of the viscosity leads to a diffusion blurring of the jump in a zone of thickness $\sim(\nu t)^{1/2}$ after a time $t$. However, the thickness of the jump zone in an ideally stratified liquid is characterized by the ratio $r_0/(Nt)^2$, as is clear from the preceding. Hence for fairly short times $(Nt \ll (Nr_0^2/\nu)^{1/5})$ the effect of viscosity on the development of the jump can be neglected. The non-linearity must play a more significant part resulting in instability of similar vorticity jumps, and by the same token to a change in the nature of the "collapse". Moreover, when passing to the limit of the vortex filament in the solution obtained $(r_0 \to 0,\ \omega_0 \to \infty,\ \omega_0 r_0^2 = \mathrm{const})$ the limit solution is found to increase with time, i.e. the basic state is unstable with respect to such singular vortex perturbations even in the linear approximation.

*Example 2.* If at the initial instant the vorticity is distributed uniformly between coaxial horizontal cylindrical surfaces

$$\omega(r, 0) = \omega_0\{h(r_1 - r) - h(r_2 - r)\}, \quad r_1 > r_2$$

then, owing to the linear formulation, the solution of the problem may be found in the form of the difference of the solutions of the previous example of two vortex filaments. Then, passing to the limit of an infinitely thin cylindrical vortex layer $(r_1 \to r_2 \to r_0,\ \Gamma_0 = \pi\omega_0(r_1^2 - r_2^2) = \mathrm{const})$

$$\omega(r, 0) = \frac{\Gamma_0}{2\pi r_0}\delta(r - r_0)$$

We express the solution of this problem in terms of the solution of the first example

$$\omega(r, t) = \frac{\Gamma_0}{4\pi r_0}\frac{\partial}{\partial r_0}\left\{r_0\int_0^\infty dk J_1(kr_0)\sum_\pm J_0(R_\pm)\right\} =$$

$$\frac{\Gamma_0}{2\pi r_0}\cos(Nt\cos\varphi)\,\delta(r - r_0) +$$

$$h(r - r_0)\frac{\Gamma_0}{2\pi r_0}\frac{\partial}{\partial r_0}\left\{\frac{2r_0^2}{r^2}\sum_{n=1} J_{2n}(Nt)P_{n-1}^{(1,0)}\left(1 - \frac{2r_0^2}{r^2}\right)\cos 2n\varphi\right\}$$

It can be seen that inside the region bounded by the vortex layer the vorticity is initially zero, and subsequently the vortex layer oscillates at constant amplitude, while in the external zone the vorticity increases (undergoing oscillations) with time. The latter can be readily checked in the special case of the vertical direction, when it is possible to use formula (2.5).

In conclusion we present two more examples with smooth non-singular vorticity distributions.

*Example 3.* For the initial distribution

$$\omega(r,0)=\omega_0\exp\left(-\frac{r^2}{2r_0^2}\right)I_0\left(\frac{r^2}{2r_0^2}\right),\quad \omega(k,0)=\omega_0\frac{2r_0\sqrt{\pi}}{k}\exp\left(-\frac{k^2r_0^2}{4}\right)$$

which in view of the properties of the modified Bessel function is close to $\omega_0$ about the axis of symmetry $(r\ll r_0)$ and falls as $\omega_0 r_0/(r\sqrt{\pi})$, while away from it $(r\gg r_0)$ the evolution of the vorticity is defined by the integral

$$\omega(\mathbf{r},t)=\frac{\omega_0 r_0}{2\sqrt{\pi}}\int_0^\infty dk\exp\left(-\frac{k^2r_0^2}{4}\right)\sum_\pm J_0(R_\pm)\tag{2.6}$$

which can be represented in the form of a Neumann series (1.5) with coefficients

$$a_{2n}(r)=\omega_0\exp\left(-\frac{r_2}{2r_0^2}\right)I_n\left(\frac{r^2}{2r_0^2}\right),\quad n=0,1,2,\ldots\tag{2.7}$$

Another convenient integral representation of the solution can be obtained, if in the formula of type (1.3) the integration is carried out not with respect to the angle but the wave number

$$\omega(\mathbf{r},t)=\frac{\omega_0}{\pi}\int_0^{\pi/2}d\theta\cos(Nt\sin\theta)\sum_\pm\exp\left\{-\frac{r^2}{r_0^2}\sin^2(\theta\pm\varphi)\right\}$$

In the case of a vertical direction by termwise integration of the exponent expansion in a power series, we obtain another Neumann series

$$\omega\Big|_{\varphi=\frac{\pi}{2}}=\frac{\omega_0}{\sqrt{\pi}}\sum_{n=0}^\infty\frac{\Gamma(n+1/2)}{n!}\left(-\frac{2r^2}{Ntr_0^2}\right)^nJ_n(Nt)\tag{2.8}$$

*Example 4.* For the initial exponential distribution

$$\omega(r,0)=\omega_0\exp\left(-\frac{r^2}{r_0^2}\right),\quad \omega(k,0)=\pi\omega_0r_0^2\exp\left(-\frac{k^2r_0^2}{4}\right)$$

the integral formula (1.4) takes the form (cf (2.6))

$$\omega(\mathbf{r},t)=\frac{\omega_0r_0^2}{4}\int_0^\infty dk\,k\exp\left(-\frac{k^2r_0^2}{4}\right)\sum_\pm J_0(R_\pm)\tag{2.9}$$

The coefficients of expansion in series of the type (1.5) are expressed in terms of modified Bessel functions of half-integral order (compare with (2.7))

$$a_{2n}(r)=\omega_0\frac{r\sqrt{\pi}}{2r_0}\exp\left(-\frac{r^2}{2r_0^2}\right)\left\{I_{n-1/2}\left(\frac{r^2}{2r_0^2}\right)-I_{n+1/2}\left(\frac{r^2}{2r_0^2}\right)\right\}$$

For the vertical direction the integral (2.9) is reduced, by changing the variable of integration $\xi=(k^2z^2+N^2t^2)^{1/2}$, to the incomplete Weber integral $Q_0(z^2/r_0^2,Nt)$ /6/.

$$\omega\Big|_{\varphi=\frac{\pi}{2}}=\omega_0\exp\left(\frac{r_0^2N^2t^2}{4z^2}-\frac{z^2}{r_0^2}\right)\left\{1-Q_0\left(\frac{z^2}{r_0^2},Nt\right)\right\}\tag{2.10}$$

$$Q_0(x,y)\equiv\frac{\exp x}{2x}\int_0^y d\xi\,\xi\,J_0(\xi)\exp\left(-\frac{\xi^2}{4x}\right),\quad Q_0(x,\infty)=1$$

In conformity with the expansion of that integral in Neumann series /6/ for the vorticity distribution along the vertical we have

$$\frac{\omega}{\omega_0}\Big|_{\varphi=\frac{\pi}{2}}=\sum_{n=0}^\infty\left(-\frac{2z^2}{r_0^2Nt}\right)^nJ_n(Nt)=\exp\left(\frac{r_0^2N^2t^2}{4z^2}-\frac{z^2}{r_0^2}\right)-\sum_{n=0}^\infty\left(\frac{r_0^2Nt}{2z^2}\right)^{n+1}J_{n+1}(Nt)\tag{2.11}$$

from which we can obtain the following asymptotic form:

$$\omega\Big|_{\varphi=\frac{\pi}{2}}\approx\omega_0\exp\left(\frac{r_0^2N^2t^2}{4z^2}-\frac{z^2}{r_0^2}\right)-\omega_0\frac{r_0^2Nt}{2z^2}J_1(Nt),\quad Nt\ll 2z^2/r_0^2$$

$$\omega\Big|_{\varphi=\frac{\pi}{2}}\approx\begin{cases}\omega_0\cos Nt/\sqrt{\pi Nt}, & Nt=2z^2/r_0^2\gg1\\ \omega_0J_0(Nt), & Nt\gg2z^2/r_0^2\end{cases}$$

Here, as in previous examples, a decisive part is played by the mixed space-time parameter $Nt\frac{r_0^2}{r^2}$.

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# STABILITY AND ADMISSIBILITY OF DISCONTINUITIES IN THE SYSTEMS OF EQUATIONS OF TWO-PHASE FILTRATION*

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To obtain the additional conditions at a discontinuity in the solution of the non-convex hyperbolic systems of equations of two-phase filtration with an active admixture /1–3/ (**) an approach is proposed that differs from the method of vanishing viscosity. The discontinuous solution is considered as the limit of solutions of the non-equilibrium system, when the characteristic time for thermodynamic equilibrium to become established approaches zero. The admissibility conditions obtained (of the existence of a structure) are the same as the equilibrium conditions in Oleinik's form /5,6/, and ensure the existence and uniqueness of the selfsimiliar solution of the problem of discontinuity disintegration.

The processes of petroleum displacement by hydrodynamically active fluids is defined by systems of non-linear differential equation of hyperbolic type, as in gas dynamics, for which discontinuous solutions are characteristic /7/. The stability of the discontinuity with respect to small perturbations is a generally acceptable requirement in the linearized problem /8,9/. However, for some non-convex sytems of the equations of gas dynamics and elasticity theory, the solution of the problem of discontinuity disintegration, containing stable discontinuities is not unique /6,10/. Supplementary conditions at the discontinuity ensuring the uniqueness of the solution were obtained either by generalizing the concept of stability, or as the limit of the solutions of the corresponding problem in a more comprehensive physical theory of "vanishing viscosity" /8–11/.

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